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THE LIMIT STATE OF A LAYER COMPRESSED BY ROUGH PLATES†

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The possibility of presenting the solution of the problem of the compression of a three-dimensional layer by two rough plates, if use is made of Prandtl's assumption of a linear variation of the shear stresses over the thickness (not depending on the coordinates along the plates), is analysed. The case of an anisotropic material with a yield point obeying Hill's condition is also considered.

The asymptotic Prandtl solution [1] for a plane layer of ideally plastic material compressed by rigid rough plates has provided the main concepts of the nature of the pressure distribution and has been used as the basis for many investigations of the theory of plastic metal forming Ivlev [2], using the condition of full plasticity of Haar and Karman [3], proposed an extension of the Prandtl solution to the case of the three-dimensional state of the layer. Below, we consider the compression of an ideally plastic layer in the case of the non-collinear direction of the contact friction on the surfaces of the compressing plates. The influence of shear stresses on the magnitude of the limit compressing force in the case of an ideally plastic anisotropic material is also investigated.

1. THE COMPRESSION OF AN IDEALLY PLASTIC ISOTROPIC LAYER

The condition of full plasticity proposed by Haar and Karman [3]

$$\sigma_1 = \sigma_2, \quad \sigma_3 = \sigma_1 + 2k, \quad k = \text{const} \tag{1.1}$$

(where σ_i are the components of the principal stresses, and k is the shear yield point) can be written in the form [4]

$$\sigma_x = \sigma - \frac{2k}{3} + 2kn_1^2, \quad \tau_{xy} = 2kn_1n_2 \ (xyz, \ 123)$$
$$n_1^2 + n_2^2 + n_3^2 = 1, \quad \sigma = \sigma_1 = \sigma_2 \tag{1.2}$$

where $\sigma_x, \tau_{xy}, \ldots$ are the components of the stresses in the Cartesian coordinates system of xyz, and n_1, n_2 and n_3 are the direction cosines determining the orientation of the third main stress σ_3 in space xyz. Here and below, the symbol $(xyz, 123)$ denotes that the relations obtained by circular permutation of the given indices should be added to the relations written.

Relations (1.2) can be rewritten in the form

$$\sigma_x = \sigma - \frac{2k}{3} + \frac{\tau_{xy}\tau_{xz}}{\tau_{yz}} \ (xyz) \tag{1.3}$$
$$\frac{\tau_{xy}\tau_{xz}}{\tau_{yz}} + \frac{\tau_{xy}\tau_{yz}}{\tau_{xz}} + \frac{\tau_{xz}\tau_{yz}}{\tau_{xy}} = 2k$$

We will consider a layer of plastic material of thickness $2h$ and introduce a Cartesian system of coordinates, so that the boundaries of the layer correspond to $z = h_1$ and $z = h_2$; $h_1 + h_2 = 2h$.

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Following Prandtl [1], we assume a linear variation of the shear stress over the layer thickness

$$\tau_{xz} = az + c_1, \quad \tau_{yz} = bz + c_2, \quad a,\, b,\, c_1,\, c_2 = \text{const} \tag{1.4}$$

From relations (1.3) and (1.4) we obtain

$$\begin{aligned}
\tau_{xy} &= \frac{(az+c_1)(bz+c_2)}{(az+c_1)^2+(bz+c_2)^2}[k \pm (k^2-(az+c_1)^2-(bz+c_2)^2)^{1/2}] \\
\sigma_x &= \sigma - \frac{2k}{3} + \tau_{xy}\frac{az+c_1}{bz+c_2} \\
\sigma_y &= \sigma - \frac{2k}{3} + \tau_{xy}\frac{bz+c_2}{az+c_1} \\
\sigma_z &= \sigma - \frac{2k}{3} + \frac{1}{\tau_{xy}}(az+c_1)(bz+c_2)
\end{aligned} \tag{1.5}$$

From the equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \ (xyz)$$

and relations (1.4) and (1.5) we obtain

$$\begin{aligned}
\sigma &= -ax - by + C + \frac{2k}{3} - \frac{\tau_{xz}\tau_{yz}}{\tau_{xy}}, \quad C = \text{const} \\
\sigma_x &= -ax - by + C + \frac{\tau_{xy}\tau_{xz}}{\tau_{yz}} - \frac{\tau_{xz}\tau_{yz}}{\tau_{xy}} \\
\sigma_y &= -ax - by + C + \frac{\tau_{xy}\tau_{yz}}{\tau_{xz}} - \frac{\tau_{xz}\tau_{yz}}{\tau_{xy}} \\
\sigma_z &= -ax - by + C
\end{aligned} \tag{1.6}$$

where τ_{xy}, τ_{xz} and τ_{yz} are defined by relations (1.4) and the first relation of (1.5).
From the last equality of (1.6) it follows that

$$\text{grad}\, \sigma_z = -a\mathbf{i} - b\mathbf{j} \tag{1.7}$$

where i and j are unit vectors along the x and y axes.
We will introduce orthogonal coordinates

$$\begin{aligned}
&\xi = x\cos\theta + y\sin\theta, \quad \eta = -x\sin\theta + y\cos\theta \\
&\cos\theta = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}
\end{aligned} \tag{1.8}$$

Then

$$\sigma_z = -\xi\sqrt{a^2+b^2} + C \tag{1.9}$$

We will give plus and minus signs respectively to the stress components on the sides of the layer. On the upper and lower sides of the layer, $z = h_1$ and $z = -h_2$, from (1.4) we obtain

$$\begin{aligned}
&\tau_{xz}^+ = ah_1 + c_1, \quad \tau_{yz}^+ = bh_1 + c_2, \quad z = h_1 \\
&\tau_{xz}^- = ah_2 + c_1, \quad \tau_{yz}^- = -bh_2 + c_2, \quad z = -h_2
\end{aligned} \tag{1.10}$$

The vectors of the shear stresses and the magnitudes of the resulting shear stresses on the sides of the layer will have the form

$$\mathbf{T}_1 = \tau_{xz}^+ \mathbf{i} + \tau_{yz}^+ \mathbf{j}, \quad \mathbf{T}_2 = \tau_{xz}^- \mathbf{i} + \tau_{yz}^- \mathbf{j} \tag{1.11}$$

$$T_1 = \sqrt{\tau_{xz}^{+2} + \tau_{yz}^{+2}} = \sqrt{(ah_1 + c_1)^2 + (bh_1 + c_2)^2} = k_1 \leqslant 1$$

$$T_2 = \sqrt{\tau_{xz}^{-2} + \tau_{yz}^{-2}} = \sqrt{(-ah_2 + c_1)^2 + (-bh_2 + c_2)^2} = k_2 \leqslant 1 \tag{1.12}$$

The angle φ between vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ and also their directions, by (1.10) and (1.11), are determined from the relations

$$\cos\varphi = \frac{\mathbf{T}_1 \cdot \mathbf{T}_2}{T_1 T_2} = \frac{c_1^2 + c_2^2 - (a^2 + b^2)h_1 h_2 + (c_1 a + c_2 b)(h_1 - h_2)}{k_1 k_2} \tag{1.13}$$

$$\operatorname{tg}\mu_1 = \frac{\tau_{yz}^+}{\tau_{xz}^+} = \frac{bh_1 + c_2}{ah_1 + c_1}, \quad \operatorname{tg}\mu_2 = \frac{\tau_{yz}^-}{\tau_{xz}^-} = \frac{bh_2 - c_2}{ah_2 - c_1}, \quad \varphi = \mu_2 - \mu_1 \tag{1.14}$$

The quantities a, b, c_1 and c_2 are determined by specifying the quantities k_1, k_2, μ_1 and μ_2.

The case where $c_1 = c_2 = 0$ and $\mu_1 = \mu_2$ was examined earlier in [2].

We will assume that the vector $\mathbf{T}_1$ is directed along the x axis. Then, from (1.10), (1.11) and (1.13), we obtain

$$\mu_1 = 0, \quad \varphi = \mu_2, \quad \cos\varphi = \frac{2ah - k_1}{k_2}, \quad \sin\varphi = \frac{2bh}{k_2}$$

From the last two equations we express a and b in terms of k_1 and k_2, and from (1.9) we obtain

$$\sigma_z = -\frac{1}{2h}(k_1^2 + k_2^2 + 2k_1 k_2 \cos\varphi)^{1/2}\xi + C \tag{1.15}$$

The equation of the line parallel to which the pressure σ_z varies, by relations (1.8), has the form

$$y = \frac{b}{a}x = \frac{k_2 \sin\varphi}{k_1 + k_2\cos\varphi}x \tag{1.16}$$

When $k_1 = k_2 = k$ we have

$$\sigma_z = -\frac{k}{h}\xi\cos\frac{\varphi}{2} + C, \quad y = \operatorname{tg}\frac{\varphi}{2}x \tag{1.17}$$

i.e. in this case the pressure σ_z increases linearly along the bisector of the angle between the directions of the stress vectors $\mathbf{T}_1$ and $\mathbf{T}_2$.

The case $\varphi = 0$ corresponds to the Prandtl solution [1]

$$\sigma_z = -\frac{k}{h}x + C, \quad \xi = x, \quad a = \frac{k}{h}, \quad c_1 = b = c_2 = 0 \tag{1.18}$$

When $\varphi = \pi/2$ we have

$$\sigma_z = -\frac{\sqrt{2}k}{h}\xi + C, \quad \xi = \frac{\sqrt{2}}{2}(x + y) \tag{1.19}$$

i.e. the pressure variation occurs along lines parallel to the line $y = x$.

To determine the constant C, it is necessary to use assumptions on the integral nature of the force distribution on the edge of the plate [5].

The relations determining the kinematics of plastic flow can be written in the form [4]

$$\varepsilon_x + \varepsilon_{xy}\frac{n_2}{n_1} + \varepsilon_{xz}\frac{n_3}{n_1} = \varepsilon_{xy}\frac{n_1}{n_2} + \varepsilon_y + \varepsilon_{yz}\frac{n_3}{n_2} = \varepsilon_{xz}\frac{n_1}{n_3} + \varepsilon_{yz}\frac{n_2}{n_3} + \varepsilon_z \tag{1.20}$$

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = 0 \tag{1.21}$$

where ε_x, ε_{xy}, ... are the components of the rate of plastic strain.

The solution is defined in the form

$$U = p_1 x + q_1 y + u(z), \quad V = p_2 x + q_2 y + v(z),$$
$$W = pz \tag{1.22}$$

where U, V and W are components of the rate of displacement.

Using expressions (1.2) and (1.21), we can give relations (1.20) the form

$$\tau_{yz}\left(\frac{2p_1}{\tau_{yz}} + \frac{q_1 + p_2}{\tau_{xz}} + \frac{u'}{\tau_{xy}}\right) = \tau_{xz}\left(\frac{q_1 + p_2}{\tau_{yz}} + \frac{2q_2}{\tau_{xz}} + \frac{v'}{\tau_{xy}}\right) = \tau_{xy}\left(\frac{u'}{\tau_{yz}} + \frac{v'}{\tau_{xz}} + \frac{2p}{\tau_{xy}}\right) \tag{1.23}$$

where the prime denotes a derivative with respect to z. The components of the shear stresses σ are determined from relations (1.5) and are functions of the z coordinate. The two equations of (1.23) define the two continuous functions, $u(z)$ and $v(z)$.

Thus, from relations (1.5), (1.22) and (1.23) and the equilibrium equations, the components of the stresses and of the rates of strain can be determined.

We will consider the case when the Prandtl solution (1.18) has superimposed upon it the shear forces

$$\tau_{xz} = \frac{k_1 z}{h}, \quad \tau_{yz} = k_2, \quad k_1^2 + k_2^2 = k^2 \tag{1.24}$$

where $k_2 = 0$ corresponds to the Prandtl solution.

From relations (1.3) and (1.24) we obtain

$$\tau_{xy} = \frac{k + \sqrt{k^2 - [(k_1 z)^2 + k_2^2]}}{(k_1 z)^2 + k_2^2} k_1 k_2 z \tag{1.25}$$

The shear forces $\tau_{yz} = k_2$ lead to the appearance of stresses τ_{xy} (1.25) determining the torsion along the y axis. The torque per unit length can be determined from the relation

$$M = 2\int_{-h_2}^{h_1} \tau_{xy} z dz$$

According to the last relation of (1.24) and (1.25), the nature of the distribution of the compressive stress

$$\sigma_z = -\frac{1}{h}\left(\sqrt{k^2 - k_2^2}\right)x + C \tag{1.26}$$

remains linear, and the shear force $\tau_{yz} = k_2$ influences the slope of the line determining the dependence of σ_z on the x coordinate. The shear force $\tau_{yz} = k_2$ also influences the value of the constant C, which is determined from the equilibrium conditions on the free edges of the layer. For the free edge of the strip $x = 0$, we must put

$$\int_{-h_2}^{h_1} \sigma_x dy = 0 \quad \text{when} \quad x = 0 \tag{1.27}$$

The stress σ_x is determined from the second relation of (1.5) and equalities (1.24).

When $k_2 = k$, by (1.24), $k_1 = 0$, and relations (1.24), (1.25) and (1.6) take the form

$$\sigma_x = C + k, \quad \sigma_y = C, \quad \tau_{xy} = \tau_{xz} = 0, \quad \tau_{yz} = k \tag{1.28}$$

From (1.27) and the first relation of (1.28) it follows that

$$C = -k \tag{1.29}$$

and the limiting stress state of the layer, when the condition of complete plasticity is satisfied, according to relations (1.1), (1.28) and (1.29), has the form

$$\sigma_x = 0, \quad \sigma_y = \sigma_z = -k, \quad \tau_{xy} = \tau_{xz} = 0, \quad \tau_{yz} = k \tag{1.30}$$

2. THE CASE OF AN ANISOTROPIC MATERIAL

For an anisotropic material, the yield point k (1.1) depends on the extension direction [2]

$$\sigma_1 = \sigma_2 = 0, \quad \sigma_3 = 2k(n_1, n_2, n_3) \tag{2.1}$$

We will assume that the surface of the yield points in xyz, space is defined in the form [6]

$$A(\sigma_x - \sigma_y)^2 + B(\sigma_y - \sigma_z)^2 + C(\sigma_z - \sigma_x)^2 + 6(F\tau_{xy}^2 + G\tau_{yz}^2 + H\tau_{xz}^2) = 6k_0^2 \tag{2.2}$$

where A, B, C, F, G, H and k_0 are constants.

From (2.1) and (2.2) we obtain

$$k^2(n_1, n_2, n_3) = \frac{3k_0^2}{2}[A(n_1^2 - n_2^2)^2 + B(n_2^2 - n_3^2)^2 + $$
$$+ C(n_3^2 - n_1^2)^2 + 6(F(n_1n_2)^2 + G(n_2n_3)^2 + H(n_3n_1)^2)]^{-1} \tag{2.3}$$

In the case of extension along the x, y or z axis, we have $n_1 = 1, n_2 = n_3 = 0$; $n_1 = n_3 = 0, n_2 = 1$; or $n_1 = n_2 = 0, n_3 = 1$ respectively. From (2.3)

$$k_1^2 = k^2(1, 0, 0) = \frac{3k_0^2}{2(A+C)} \quad (123, ABC)$$
$$k_2^2 = k^2(0, 1, 0), \quad k_3^2 = k^2(0, 0, 1) \tag{2.4}$$

We will then have

$$k_4^2 = k^2\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) = \frac{6k_0^2}{B+C+6F} \quad (456, ABC, FGH) \tag{2.5}$$

From relations (2.4) and (2.5) we find

$$A = \frac{3}{4}k_0^2\left(\frac{1}{k_1^2} + \frac{1}{k_2^2} - \frac{1}{k_3^2}\right) (123, ABC) \quad F = k_0^2\left(\frac{1}{k_4^2} - \frac{1}{4k_3^2}\right) (123, 456, FGH) \tag{2.6}$$

According to relations (2.6), the form of the limit surface is determined entirely by the yield points k_1, k_2, k_3, k_4, k_5 and k_6. The yield point $k(n_1, n_2, n_3)$ in any direction, determined by n_1, n_2 and n_3 according to (2.3)–(2.7), is determined by setting the yield points k_i, $i = 1, 2, \ldots, 6$.

From relations (1.3) and (2.2) we find

$$\tau_{xy}^4\left[(A+B)\left(\frac{\tau_{yz}}{\tau_{xz}}\right)^2 + (A+C)\left(\frac{\tau_{xz}}{\tau_{yz}}\right)^2 - 2A + 6F\right] -$$
$$-2\tau_{xy}^2[(B-3G)\tau_{yz}^2 + (C-3H)\tau_{xz}^2 + 3k_0^2] + (B+C)\tau_{xz}^2\tau_{yz}^2 = 0 \tag{2.7}$$

Adopting assumption (1.4), from (1.5) and (2.7) we have

$$\tau_{xy} = \tau_{xy}(z). \tag{2.8}$$

According to relations (1.3), (1.4) and (2.8), the quantity $k = k(n_1, n_2, n_3)$ is a function of the single variable z. Consequently, equalities (1.3) and (1.4) and the equilibrium equations lead to the last three expressions of (1.6). The magnitude of the compressive pressure will be determined according to the last equality of (1.6)

$$\sigma_z = -ax - by + C \tag{2.9}$$

The form of the anisotropy has no influence on the nature of the pressure distribution, determined by σ_z. In expressions for the components σ_x and σ_y (1.6), the influence of the anisotropy, according to (1.4) and (2.7), is determined by τ_{xy} (2.7), and the influence of the anisotropy appears in the magnitude of the constant C, (1.6) and (2.9), which is determined from the boundary conditions.

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